*International Journal of Theoretical Physics*, Vol. 14, No. 6 (1975), pp. 423–428

# Unstable Particles in a Self-Consistent Field Theory

M. SEETHARAMAN and V. SRINIVASAN[1]

*Department of Theoretical Physics, University of Madras, Madras-25, India*

*Received: 8 April 1975*

*Abstract*

We indicate how unstable particles can be introduced into the self-consistent field theory formulation of Umezawa where the equal-time commutation relations for Heisenberg fields are derived and not assumed. The Lee model is used to illustrate the results.

## 1. *Introduction*

In an earlier paper one of the authors (VS) with Rest and Umezawa (Rest et al., 1971) illustrated the main computational steps in a self-consistent formulation (Umezawa, 1965) of quantum field theory in which one uses only the Fock space of physical particles. The equal-time commutation relations (ETCR) for Heisenberg fields are derived in this formulation in a self-consistent way (Rest et al., 1971; Seetharaman & Srinivasan, 1975). The field equations together with the requirement of microcausality will show whether the initial choice of the physical fields is correct or not. For instance, it was shown for the pair model (Rest et al., 1971) that a separate field for the physical $N$-$\theta$ bound state (in addition to the physical $N$ and $\theta$ fields) is necessary for microcausality. In the present short paper we indicate and demonstrate through a solvable model how unstable particles (Levy, 1959; Glaser & Kallen, 1956; Schulman, 1970) can be incorporated into this scheme. Though vector spaces for unstable particles have been considered in the literature (Hammer & Weber, 1972), we emphasize that in this method there is no need to go beyond the conventional Fock space of physical particles.

In the next section we mention briefly the main assumptions of the self-consistent method and point out how composite and/or unstable particles are introduced. In the last section we illustrate the results through the Lee model. In order to save space we shall keep all calculations to the barest minimum and refer the reader to two earlier papers (Rest et al., 1971; Seetharaman

[1] C.S.I.R. Pool Officer.

& Srinivasan, 1975) where the method of calculation has been outlined in great detail.

## 2. *Unstable Particles in the Self-Consistent Method*

We shall start with a brief enumeration of the important postulates of the method:

i. Equations of motion for the interacting Heisenberg fields $\psi^{(i)}$ $(i, = 1, \ldots, n)$ are assumed given.

ii. With a proper choice of the physical fields $\phi^{(j)}$ $(j = 1, \ldots, m)$ a map (known as the dynamical map, Laplae et al., 1965) is constructed in which one expands the Heisenberg fields in terms of the normal products of the physical fields with expansion coefficients to be determined from the field equations. Any invariance properties of the field equations are of great help in writing down such a map (Laplae et al., 1974). It is important to note that the number $m$ of physical fields *need not always be equal to* the number $n$ of Heisenberg fields occurring in the equations of motion. This is a point which we shall amplify below. The Fock space for the physical fields is constructed in the usual way.

iii. The $\phi^{(i)}$-*s* are taken to be the "in-fields" and hence the coefficients of the map are assumed retarded in nature.[2]

iv. The Heisenberg fields obey microcausality. There exists a unitary transformation $S$ that transforms $a^{\text{in}}$ into $a^{\text{out}}$

$$a^{\text{out}} = S^{-1} a^{\text{in}} S$$

Here $a^{\text{in}}$ $(a^{\text{out}})$ is the usual Fourier coefficient (annihilation operator) of the physical field $\phi^{\text{in}}$ $(\phi^{\text{out}})$.

The compositeness or instability of particles in this method is linked to the second postulate. If we have $n$ Heisenberg fields (satisfying certain postulated equations) and $m$ physical fields (satisfying free field equations with physical masses) which form a complete set and if $m < n$ then we have $(m - n)$ composite particles in the theory. In the pair model (Rest et al., 1971) one started with two Heisenberg fields, but at the "in" level three physical fields were found necessary for microcausality and thus one identified one of the fields as composite. On the contrary if $m < n$, i.e., the number of physical fields that form a complete set is less than the number of Heisenberg fields, then we have $(n - m)$ unstable particles. Using the postulates it can also be easily shown that the unstable particle defined this way has no asymptotic field.[3]

## 3. *Lee Model with Unstable V Particle*

We postulate the following set of equations of the Lee-Dirac model:

$$\left(\frac{\partial}{\partial t} + im_0\right)\psi(y) = -ig\int d^4x\alpha(x - y)\theta^\dagger(x)V(y) \tag{3.1}$$

[2] This is the asymptotic condition.

[3] The proof of this just involves the use of the Reimann–Lebesgue theorem.

$$\left(\frac{\partial}{\partial t} + i\sqrt{\mu_0^2 - \nabla^2}\right)\theta(y) = -ig\int d^4x\alpha(y-x)\psi^\dagger(x)V(x) \tag{3.2}$$

$$\left(\frac{\partial}{\partial t} + iM_0\right)V(y) = -ig\int d^4x\alpha(x-y)\theta(x)\psi(y) \tag{3.3}$$

Here the cut-off function $\alpha(x)$ is given by

$$\alpha(x-y) = \delta(t_x - t_y)\frac{1}{(2\pi)^{3/2}}\int d^3k\,\frac{\alpha(\omega_{\mathbf{k}})}{(2\omega_{\mathbf{k}})^{1/2}}\,e^{i|\mathbf{k}||\mathbf{x}-\mathbf{y}|} \tag{3.4}$$

with $\omega_{\mathbf{k}} = (\mathrm{k}^2 + \mu_{\mathbf{k}}^2)^{1/2}$.

We shall consider the $V$ particle to be unstable.[4] As explained earlier, that means that, in the dynamical map, the $V^{\text{in}}$ field does not occur. The mapping for this can easily be found to be

$$\begin{aligned}\psi(x) = \psi^{\text{in}}(x) + \int d^3p\,d^3q\,d^3r\,c_{\mathbf{p}}(\mathbf{q},\mathbf{r})\theta_{\mathbf{q}}^{\text{in}\dagger}N_{\mathbf{p}+\mathbf{q}-\mathbf{r}}^{\text{in}}\theta_{\mathbf{r}}^{\text{in}}\\ \times e^{i\mathbf{p}\cdot\mathbf{x}}e^{-i(\omega_{\mathbf{r}}-\omega_{\mathbf{q}}+m_{\mathbf{p}+\mathbf{q}-\mathbf{r}})t}+\cdots\end{aligned} \tag{3.5}$$

$$\begin{aligned}\theta(x) = \theta^{\text{in}}(x) + \int d^3p\,d^3q\,d^3r\,g_{\mathbf{p}}(\mathbf{q},\mathbf{r})N_{\mathbf{q}}^{\text{in}\dagger}N_{\mathbf{p}+\mathbf{q}-\mathbf{r}}^{\text{in}}\theta_{\mathbf{r}}^{\text{in}}\\ \times e^{i\mathbf{p}\cdot\mathbf{x}}e^{-i(m_{\mathbf{p}+\mathbf{q}-\mathbf{r}}+\omega_{\mathbf{r}}-m_{\mathbf{q}})t}+\cdots\end{aligned} \tag{3.6}$$

$$V(x) = \int d^3p\,d^3q\,N_{\mathbf{p}-\mathbf{q}}^{\text{in}}\theta_{\mathbf{q}}^{\text{in}}h_{\mathbf{p}}(\mathbf{q})e^{i\mathbf{p}\cdot\mathbf{x}}e^{-i(\omega_{\mathbf{q}}+m_{\mathbf{p}+\mathbf{q}})t}+\cdots \tag{3.7}$$

Here

$$\psi^{\text{in}}(x) = \frac{1}{(2\pi)^{3/2}}\int d^3k\,N_{\mathbf{k}}^{\text{in}}e^{i(\mathbf{k}\cdot\mathbf{x}-m_{\mathbf{k}}t)} \tag{3.8}$$

$$\theta^{\text{in}}(x) = \frac{1}{(2\pi)^{3/2}}\int d^3k\,\theta_{\mathbf{k}}^{\text{in}}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega_{\mathbf{k}}t)} \tag{3.9}$$

and we assume

$$[N_{\mathbf{p}}^{\text{in}}, N_{\mathbf{q}}^{\text{in}\dagger}]_+ = [\theta_{\mathbf{p}}^{\text{in}}, \theta_{\mathbf{q}}^{\text{in}\dagger}] = \delta(\mathbf{p}-\mathbf{q}) \tag{3.10}$$

We now have to determine all the unknowns in the above map and then check whether microcausality for the Heisenberg fields is satisfied. The map coefficient $c_{\mathbf{p}}, g_{\mathbf{p}}, h_{\mathbf{p}}$, together with the masses of the physical fields are determined by considering the matrix elements of the Heisenberg fields between

[4] It is not necessary to assume that the mass of the unstable particle $M > m + \mu$ at this stage. See below.

physical "in" states. We shall omit all details (see Seetharaman & Sriniyasan, 1975) and give the results below:

$$m_1 = m_0 = m, \qquad \mu_1 = \mu_0 = \mu$$

$$c_{\mathbf{u}+\mathbf{t}-\mathbf{s}}(\mathbf{s}, \mathbf{t}) = \frac{1}{(2\pi)^{3/2}}\left(\frac{\omega_{\mathbf{u}}}{\omega_{\mathbf{s}}}\right)^{1/2} \frac{\alpha(\omega_{\mathbf{s}})}{\alpha(\omega_{\mathbf{u}})} \frac{f(\omega_{\mathbf{u}})}{\omega_{\mathbf{u}} - \omega_{\mathbf{s}} + i\epsilon}$$

$$g_{\mathbf{u}+\mathbf{t}-\mathbf{s}}(\mathbf{s}, \mathbf{t}) = \frac{1}{(2\pi)^{3/2}}\left(\frac{\omega_{\mathbf{t}}}{\omega_{\mathbf{t}+\mathbf{u}-\mathbf{s}}}\right)^{1/2} \frac{\alpha(\omega_{\mathbf{u}+\mathbf{t}-\mathbf{s}})}{\alpha(\omega_{\mathbf{t}})} \frac{f(\omega_{\mathbf{t}})}{\omega_{\mathbf{t}} - \omega_{\mathbf{u}+\mathbf{t}-\mathbf{s}} + i\epsilon}$$

$$h_{\mathbf{t}+\mathbf{u}}(\mathbf{u}) = \frac{1}{(2\pi)^{3/2}} \frac{f(\omega_{\mathbf{u}})(2\omega_{\mathbf{u}})^{1/2}}{g\alpha(\omega_{\mathbf{u}})}$$

with

$$f(\omega_{\mathbf{u}}) = g' \frac{\alpha^2(\omega_{\mathbf{u}})}{2\omega_{\mathbf{u}}} \frac{1}{1 - g' \pm (\omega_{\mathbf{u}})}$$

and

$$g' = \frac{g^2}{\omega_{\mathbf{u}} + m - M_0}, \qquad I(\omega_{\mathbf{u}}) = \int \frac{d^3k}{2\omega_{\mathbf{k}}} \frac{\alpha^2(\omega_{\mathbf{k}})}{\omega_{\mathbf{u}} - \omega_{\mathbf{k}} + i\epsilon}$$

The $i\epsilon$ factor in the denominator is due to the retarded nature of the coefficients of the dynamical map.

We now check whether the dynamical map we have obtained is consistent with microcausality. For this consider

$$\langle N^{\text{in}}_{\mathbf{l}+\mathbf{k}} | [\psi^+(x), \theta(y)]_{tx=ty} | \theta^{\text{in}}_{\mathbf{k}} \rangle$$

Evaluating this we find that

$$\left[\frac{1}{(2\pi)^{3/2}} (g_{\mathbf{p}}(\mathbf{l} + \mathbf{k}, \mathbf{k}) + c^*_{\mathbf{p}+\mathbf{l}}(\mathbf{k}, \mathbf{l} + \mathbf{k})) + \int d^3r g_{\mathbf{p}}(\mathbf{l} + \mathbf{k}, \mathbf{r}) c^*_{\mathbf{p}+\mathbf{l}}(\mathbf{k}, \mathbf{p} + \mathbf{l} + \mathbf{k} - \mathbf{r})\right]$$

must be a finite polynomial in **p** (say $J$) for microcausality. Using the computed values of coefficients and the fact that

$$d^3r = 4\pi(\omega^2 - \mu^2)^{1/2} \omega d\omega$$

and

$$f^*(\omega) - f(\omega) = 8\pi^2 i |f(\omega)|^2 \omega(\omega^2 - \mu^2)^{1/2}$$

we find $J$ to be

$$J = A(\mathbf{p}, \mathbf{k})(\omega_{\mathbf{p}} - \omega_{\mathbf{k}}) I$$

with

$$A(\mathbf{p}, \mathbf{k}) = \frac{\alpha(\omega_{\mathbf{k}})\alpha(\omega_{\mathbf{p}})}{(2\pi)^{3/2}(4\omega_{\mathbf{p}}\omega_{\mathbf{k}})^{1/2}} \frac{1}{\omega_{\mathbf{k}} - \omega_{\mathbf{p}} + i\epsilon}$$

and

$$I_1(\omega_p, \omega_k) = \int_C \frac{g'}{1 - g'I(\omega)} \frac{d\omega}{(\omega - \omega_p)(\omega - \omega_k)}$$

Here $C$ is the contour in the $\omega$ plane which encloses a cut along the positive real axis from $\omega = \mu$ to $\omega = \infty$. If the contour encloses a pole then it is easily seen that $J$ is not a finite polynomial and hence microcausality will be violated. If, however, there is no pole in the cut plane than $I_1 = 0$ and hence microcausality is preserved. This means that $1 - g'I(\omega) \neq 0$ in the cut plane, that is the coupling constant is such that there is no pole on the real axis in cut plane. So if one were to give a mass to the unstable particle, the mass equation would read as $1 - g'I(M - m) = 0$ implying $M > m + \mu$ (the pole is in the branch cut). The unstable particle has a complex mass. One can now compute the $S$ matrix and indeed verify that the unstable particles appear as a pole in the second Reimann sheet. The ETCR for the Heisenberg fields can now be computed, after verifying that microcausality is satisfied in all sectors for the map obtained, and they are found to be the same as in the canonical field theory.

One point deserves special mention. In field theory there has been an attractive hypothesis due to Jouvet (Jouvet, 1956; Whippman, 1971; Lurie, 1968) of the vanishing of the wave function renormalization constant for identifying an elementary particle as a bound state. Our calculation seems to indicate that this conjecture could be sharpened. A particle is deemed to be a bound state if $z \to 0$ but its field appears in the physical set of fields. The dynamical map together with Heisenberg equations give us the mass of the bound state. On the other hand if the particle is unstable, again $z \to 0$ but in addition no field appears for the particle at the level of physical fields. Of course the microscopic causality condition plays a crucial role in deciding whether the particle is unstable or bound in a given theory.

*Acknowledgment*

The authors wish to thank Professor P. M. Mathews for encouragement and useful comments.

## *References*

Glaser, V. and Kallen, G. (1956). *Nuclear Physics*, **2**, 706.
Hammer, C. L. and Weber, T. A. (1972). *Physical Review*, **5**, 3087.
Jouvet, B. (1956). *Nuovo Cimento*, **3**, 1133.
Laplae, L., Sen, R. N. and Umezawa, H. (1965). *Progress in Theoretical Physics Supplements*, Commemoration Issue for the 30th Anniversary of the Meson theory by Dr. H. Yukawa.
Laplae, L., Mancini, F. and Umezawa, H. (1974). *Physics Reports*, **10c**.
Levy, H. (1959). *Nuovo Cimento*, **13**, 115; **14**, 612.

Lurie, D. (1968). *Particles and Fields,* pp. 444–446. Interscience, New York.
Rest, J., Srinivasan, V., and Umezawa, H. (1971) *Physical Review D,* **3**, 1890.
Schulman, L. S. (1970). *Annals of Physics,* **59**, 201.
Seetharaman, M. and Srinivasan, V. (1975). *Progress in Theoretical Physics,* **53**, March issue.
Umezawa, H. (1965). *Acta Physics Hungary Temp.,* **19**, 9.
Whippman, X (1971).